Evidence for universality within the classes of deterministic and stochastic sandpile models

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(Received 16 October 2000; published 23 May 2001)

Recent numerical studies have provided evidence that within the family of conservative, undirected sandpile models with short range dynamic rules, deterministic models such as the Bak-Tang-Wiesenfeld model [P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987)] and stochastic models such as the Manna model [S. S. Manna, J. Phys. A **24**, L363 (1991)] belong to different universality classes. In this paper we examine the universality within each of the two classes in two dimensions by numerical simulations. To this end we consider additional deterministic and stochastic models and use an extended set of critical exponents, scaling functions, and geometrical features. Universal behavior is found within the class of deterministic Abelian models, as well as within the class of stochastic models (which includes both Abelian and non-Abelian models). In addition, it is observed that deterministic but non-Abelian models exhibit critical exponents that depend on a parameter, namely they are nonuniversal.

DOI: 10.1103/PhysRevE.63.061309 PACS number(s): 05.70.Jk, 05.40.−a, 05.70.Ln

I. INTRODUCTION

Sandpile models were introduced over a decade ago as a paradigm of self-organized criticality (SOC) [1–3]. SOC provides a useful framework for the study of driven nonequilibrium systems that dynamically evolve into a critical state. At the critical state these systems exhibit avalanche dynamics with long-range spatial and temporal correlations, which resemble the behavior at equilibrium critical points. In sandpile models, defined on a lattice, grains are deposited randomly until the height at some site exceeds some threshold, thus becoming unstable. Grains from the unstable site are distributed between its nearest neighbors, which may become unstable too, resulting in an avalanche. These models were found to be self-driven into a critical state in which the avalanche sizes follow a power-law distribution. The critical state, which can be characterized by various critical exponents and scaling functions, was studied extensively using both theoretical [4–9] and numerical approaches [10–16].

To examine the dependence of the critical state on various properties of the models, different sandpile models have been introduced such as the Manna [17] and the Zhang [18] models. The issue of universality has been debated. Analytical studies [19–23] and numerical simulations [16,24] indicated that the Manna model, which is stochastic, as well as the Zhang model, which is deterministic and non-Abelian, belong to the universality class of the original model introduced by Bak, Tang, and Wiesenfeld (BTW), which is deterministic and Abelian [6]. Numerical simulations using an extended set of critical exponents provided evidence that deterministic Abelian and stochastic models exhibit different scaling properties and thus belong to different universality classes [25,26]. Further support for this hypothesis was recently obtained using multifractal analysis [27], moment analysis [28], as well as studies of sandpile models as closed systems [29–31].

In this paper we examine the universality within the class of deterministic Abelian models and within the class of stochastic models in the two-dimensional case. To this end we consider additional deterministic Abelian and stochastic models and examine their scaling properties using numerical simulations and an extended set of critical exponents, scaling functions, and geometrical features. We obtain evidence for universal behavior within each of the two classes as well as further evidence that these classes are different from each other.

The paper is organized as follows. The models are introduced in Sec. II. The simulations and results are presented in Sec. III, followed by a discussion in Sec. IV, and a summary in Sec. V.

II. MODELS

A. General definitions and properties

Sandpile models are defined on a d-dimensional lattice of linear size L. Each site $\mathbf{i}$ is assigned a dynamic variable $E(\mathbf{i})$ that represents some physical quantity such as energy, grain density, stress, etc. A configuration $\{E(\mathbf{i})\}$ is called *stable* if for all sites $E(\mathbf{i})<E_c$, where E_c is a threshold value. The evolution between stable configurations is by the following rules. (i) Adding energy: Given a stable configuration $\{E(\mathbf{j})\}$ we select a site $\mathbf{i}$ at random and increase $E(\mathbf{i})$ by some amount δE. When an unstable configuration is reached, rule (ii) is applied. (ii) Relaxation (or toppling) rule: If $E(\mathbf{i}) \geqslant E_c$, relaxation takes place and energy is distributed in the following way:

$$E(\mathbf{i})\rightarrow E(\mathbf{i})-\sum_{\mathbf{e}} \Delta E(\mathbf{e}),$$

$$E(\mathbf{i}+\mathbf{e})\rightarrow E(\mathbf{i}+\mathbf{e})+\Delta E(\mathbf{e}), \tag{1}$$

where $\mathbf{e}$ are a set of vectors from the site $\mathbf{i}$ to some neighbors. As a result of the relaxation, $E(\mathbf{i}+\mathbf{e})$ for one or more of the neighbors may exceed the threshold E_c. The relaxation rule is then applied until a stable configuration is reached. The sequence of relaxations is an avalanche that propagates through the lattice. Avalanches can be characterized by properties such as their size, area, and lifetime. The size s of an avalanche is the total number of relaxation events that occurred during the course of the avalanche. The area a is the number of lattice sites that experienced at least one relaxation event during the avalanche. In order to obtain a sensible definition of the avalanche lifetime t, we first need to define the time step. A single *time-step* iteration is defined as the relaxation of all the sites that satisfied $E(\mathbf{i})\geqslant E_c$, after the previous iteration was completed. Then, the lifetime of the avalanche is defined as the number of time steps that took place during the avalanche.

The critical exponents depend on the vector ΔE, to be termed *relaxation vector*. For a square lattice with relaxation to nearest neighbors it is of the form $\Delta E = (E_N, E_E, E_S, E_W)$, where E_N, E_E, E_S, and E_W are the amounts transferred to the northern, eastern, southern, and western nearest neighbors, respectively. Using the relaxation vector we can characterize the sandpile models as either *deterministic* or *stochastic*. If the toppling rule, given by ΔE is a constant or is given by a deterministic function of $E(\mathbf{i})$ for the toppling site $\mathbf{i}$ then the model is deterministic. Any stochastic component in ΔE makes the model stochastic. Another distinction is between *Abelian* and *non-Abelian* models [6,32]. Consider a stable initial configuration and a series of two avalanches initiated by adding energy δE to sites 1 and 2. The model is Abelian if the resulting stable configuration after the two avalanches is independent of the order in which the energy was added to 1 and 2 as well as of the order in which unstable sites are toppled. Any dependence on the order makes the model non-Abelian.

The sandpile models considered in this paper are conservative in the sense that the dynamics conserves energy. These models can be studied either with closed (or periodic) boundary conditions, where the total energy is fixed, or with open boundary conditions. For open boundaries, when an avalanche reaches a boundary site some energy is transferred out of the system (namely, dissipation takes place at the boundaries). In the case of open boundary conditions the critical state is reached spontaneously in the limit in which the random addition of energy (or drive) is infinitely slow (practically it means that the next energy unit is added only after the previous avalanche is completed). This state is characterized by a power-law distribution of avalanche sizes. In the critical state the added energy δE per avalanche is balanced on average by the energy that flows out through the boundaries. Therefore, the average amount of energy leaving the system per avalanche is δE.

B. Previously introduced models

In the BTW model, $E_c=4$, $\delta E=1$, and $\Delta E=(1,1,1,1)$. Since ΔE is a constant this model is clearly deterministic. Note that since ΔE is independent of $E(\mathbf{i})$, if an active site with $E(\mathbf{i})>E_c$ is toppled, it remains nonempty after the toppling event had occurred. The BTW model was shown to belong to a class of Abelian models [6]. This can be easily demonstrated by considering two nearest neighbor sites 1 and 2 that are active simultaneously. Although the site that topples first adds a unit of energy to the other site, that site still distributes only four units, independently of how supercritical it became.

The Zhang model is a deterministic model for which $E_c=1$ and δE is chosen as a constant value in range $0<\delta E<1$ [18]. The relaxation vector is given by (b,b,b,b), where $b=E(\mathbf{i})/4$ and $E(\mathbf{i})$ is the amount of energy in the active site before the toppling had occurred. Clearly, the site $\mathbf{i}$ remains empty after toppling. It is easy to see that the Zhang model is non-Abelian. Consider two nearest neighbor sites 1 and 2 that are active simultaneously, namely, $E(1)\geqslant E_c$ and $E(2)\geqslant E_c$. If we first topple site 1 and then topple site 2 (assuming no further sites becomes unstable), in the resulting configuration $E(1)>0$ and $E(2)=0$. If the order of topplings is reversed one obtains $E(1)=0$ and $E(2)>0$, namely, the resulting configuration depends on the order of topplings and thus the model is non-Abelian. The critical exponents of the Zhang model were found to be independent of the value of the added energy δE as long as $\delta E \ll E_c$ [16].

In a class of stochastic sandpile models, first introduced by Manna, a set of neighbors is randomly chosen for relaxation [17]. Such models can be specified by a set of relaxation vectors, each vector being assigned a probability for it application. For example, a possible realization of a Manna two-state model, with $E_c=2$ and $\delta E=1$, includes six relaxation vectors (1,1,0,0), (1,0,1,0), (1,0,0,1), (0,1,1,0), (0,1,0,1), and (0,0,1,1), each one applied with a probability of 1/6. In the original model introduced by Manna [17] an unstable site distributes all its energy to its neighbors and becomes empty. It can be shown that this model is non-Abelian using the argument presented above for the Zhang model. Later, various Abelian versions of the Manna model were introduced, in which the number of energy units distributed from an unstable site is a constant (see, e.g., Refs. [25,33,31]). To see whether these models are Abelian, consider two simultaneously unstable nearest neighbor sites 1 and 2. Assume that the relaxation vectors of these sites are predetermined (randomly) and are independent of the order of their toppling. In this case, just like in the BTW model, the resulting configuration is independent of the order of topplings. Note however, that unlike the BTW model in which the final configuration is fully determined by the initial unstable configuration, in the Manna model the final configuration depends also on the random choice of the relaxation vectors. Another property of the Manna model is that the relaxation vector is isotropic only on average, while the single topplings are anisotropic.

C. Additional models

We will now present additional models to be used for the study of universality within the classes of deterministic and stochastic models. These models are variants of the Zhang model. In the *Generalized Zhang* model the Zhang relaxation vector is modified into $\Delta E=(b,b,b,b)$ where $b=pE(\mathbf{i})/4$ and $0<p\leqslant 1$ is a predefined constant [34,35]. In this model only a fraction p of the energy in the unstable site $\mathbf{i}$ is distributed to its neighbors, and the rest remains in $\mathbf{i}$. Like the Zhang model, this model is deterministic and non-Abelian. Consider a pair of nearest neighbor sites 1 and 2 for which $E(1)\geqslant E_c$ and $E(2)\geqslant E_c$ simultaneously. If we first topple site 1 and later topple site 2 we obtain

$$E(1)\rightarrow(1-p)E(1)+\frac{p}{4}E(2)+\frac{p^2}{16}E(1),$$

$$E(2)\rightarrow(1-p)E(2)+\frac{p}{4}E(1)-\frac{p^2}{4}E(1),$$

$$E(1+\mathbf{e})\rightarrow E(1+\mathbf{e})+\frac{p}{4}E(1),$$

$$E(2+\mathbf{e})\rightarrow E(2+\mathbf{e})+\frac{p}{4}E(2)+\frac{p^2}{16}E(1). \qquad (2)$$

Reversing the order of topplings of these two sites we obtain

$$E(1)\rightarrow(1-p)E(1)+\frac{p}{4}E(2)-\frac{p^2}{4}E(2),$$

$$E(2)\rightarrow(1-p)E(2)+\frac{p}{4}E(1)+\frac{p^2}{16}E(2),$$

$$E(1+\mathbf{e})\rightarrow E(1+\mathbf{e})+\frac{p}{4}E(1)+\frac{p^2}{16}E(2),$$

$$E(2+\mathbf{e})\rightarrow E(2+\mathbf{e})+\frac{p}{4}E(2). \qquad (3)$$

The difference between the two resulting configurations is found to be of order p^2. Therefore, in the limit $p\rightarrow 0$ this difference vanishes and the order of relaxations becomes irrelevant. The model becomes Abelian in this limit and is termed the *Abelian Zhang* model. This limit resembles the situation in rotations of a rigid body in three-dimensional space, where infinitesimal rotations are Abelian while finite rotations are non-Abelian [36]. Moreover, within this analogy, the BTW model may correspond to the group of rotations by 180° around the x, y, and z axes, which is also Abelian.

We also consider another modification of the Zhang model, in which $\Delta E=(b,b,b,b)$, where $b=pE(\mathbf{i})/4$ and for each toppling event p is chosen randomly in the range $0<p\leqslant 1$. This model, which is non-Abelian and stochastic is termed *stochastic Zhang* model.

III. SIMULATIONS AND RESULTS

To characterize the models described above we have performed extensive computer simulations and calculated an extended set of characterization measures for two-dimensional sandpile models. These measures include the distribution exponents, the geometric exponents, as well as scaling functions and geometric features of the avalanches [26]. The *distribution exponent* τ_x characterizes the distribution of the avalanche parameter x. It is found that $P(x)\sim x^{1-\tau_x}$, where x may represent the avalanche size s, area a, or lifetime t. The *geometric exponent* γ_{xy} relates the distributions of x and y and is defined in terms of the conditional expectation value $E[x|y]\sim y^{\gamma_{xy}}$ where $x,y\in\{s,a,t\}$ [13,14]. The geometric exponents satisfy the relations

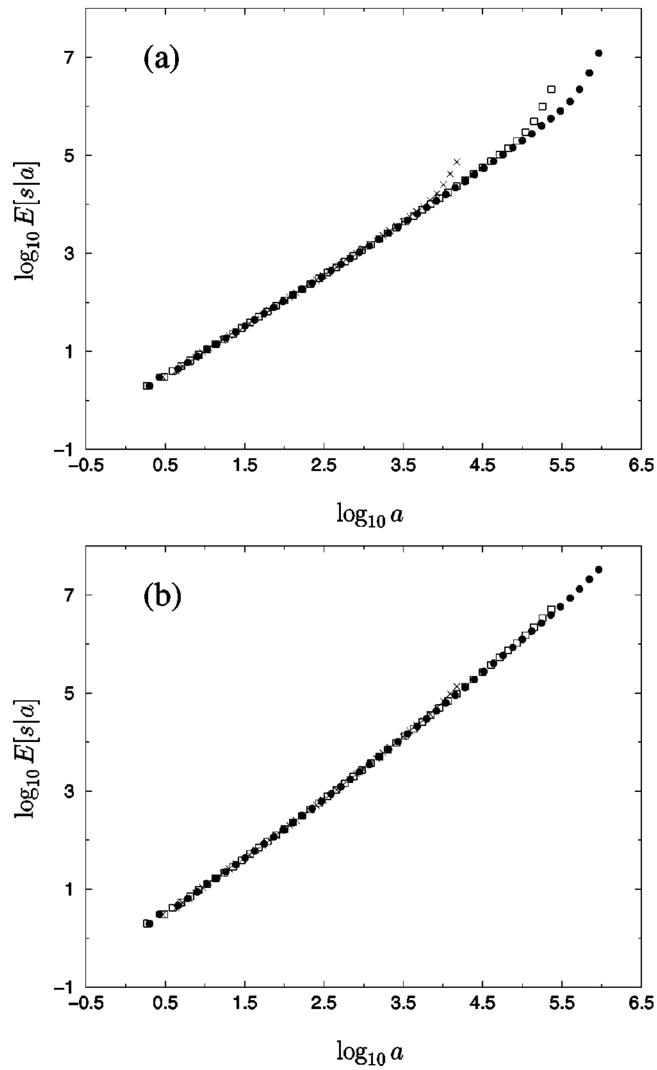

FIG. 1. The conditional expectation values $E[s|a]$ vs a for the BTW model (a) and for the Manna two-state model (b) for three different system sizes: 128(×), 512(□), and 1024(●). The exponent γ_{sa}, given by the slope in the straight segment is $\gamma_{sa}=1.06\pm 0.01$ for the BTW model and $\gamma_{sa}=1.23\pm 0.01$ for the Manna model.

$$\gamma_{yx}=\gamma_{xy}^{-1} \qquad (4)$$

and

$$\gamma_{zx}=\gamma_{zy}\gamma_{yx}. \qquad (5)$$

The geometric exponent γ_{sa} is studied by drawing on a log-log scale $E[s|a]$ vs a, where $E[s|a]$ is the conditional expectation value of the avalanche size for avalanches of a given area a. It is shown for system sizes of $L=128$, 512, and 1024 for the BTW model [Fig. 1(a)] and for the Manna model [Fig. 1(b)]. For both models the lines coincide in the common range of scales due to the fact that the geometric exponents are weakly dependent on the system size [25,26]. The slope of the straight segments gives rise to $\gamma_{sa}=1.06\pm 0.01$ for the BTW model and $\gamma_{sa}=1.23\pm 0.01$ for the Manna model.

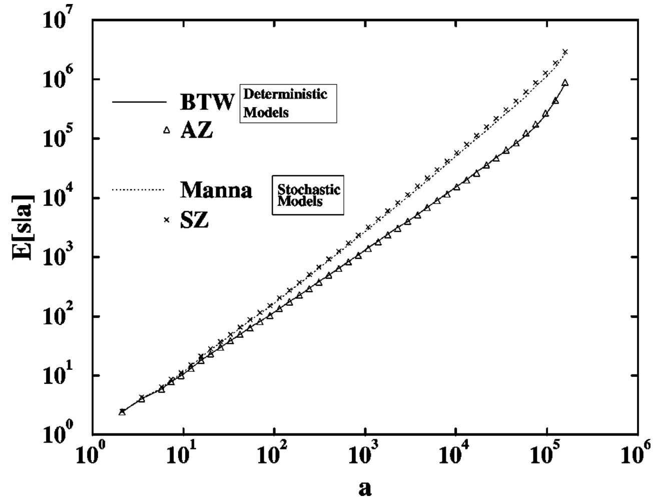

FIG. 2. The conditional expectation values $E[s|a]$ vs a for the deterministic Abelian models (BTW and Abelian Zhang) and for the stochastic models (Manna and stochastic Zhang). Clearly, the $E[s|a]$ values for the Abelian Zhang model coincide with those obtained for the BTW model, while the stochastic Zhang results coincide with those obtained for the Manna model. This indicates that while there is a clear distinction between the deterministic Abelian and stochastic models, within each of the two classes γ_{sa} is universal. The system size used here is $L=512$ and the parameter p used in the simulation of the Abelian Zhang model is $p=0.005$.

The conditional expectation values $E[s|a]$ vs a are shown in Fig. 2 for the BTW and Abelian Zhang models that are deterministic as well as for the Manna and stochastic Zhang models that are stochastic. For the deterministic models we find that the exponent γ_{sa} for the Abelian Zhang model coincides with its value for the BTW model. For the stochastic models we find that the exponent γ_{sa} for the stochastic Zhang model coincides with its value for the Manna model. These results indicate that there is a considerable degree of universality within each of the two classes.

The avalanche properties studied above, such as the area a and size s, characterize an avalanche as a whole. These properties are measured only after the avalanche is completed. To examine the evolution during the avalanche one can study the rate of change of a and s during the avalanche. The avalanche area $a_c(t)$ is defined as the number of sites where at least one relaxation occurred during the first t time steps of the avalanche. As the avalanche is completed $a_c(t)$ coincides with the area a of the avalanche. We define $a(t)$ as the time derivative of $a_c(t)$, namely, $a(t)=da_c(t)/dt$ where $da_c(t)/dt \equiv a_c(t+1)-a_c(t)$. The variable $a(t)$ gives the number of sites that at time t became active for the first time (and are to be toppled in the next time step). As the avalanche evolves to an end we find that the avalanche area is given by $a=\Sigma_{t=0}^{t_{max}} a(t)$, where t_{max} is the avalanche lifetime, and the $t=0$ step consists of the deposition event that initiated the avalanche. Similarly $s(t)$ is defined as the number of active sites at time t (namely, sites that are to topple in the next time step). Then, the avalanche size is given by $s=\Sigma_{t=0}^{t_{max}} s(t)$. When $a(t)$ and $s(t)$ are plotted for a single avalanche, highly irregular functions are obtained in the range $0 \leq t \leq t_{max}$ (note that for an avalanche of lifetime t_{max}: $a(t)=s(t)=0$ for $t \geq t_{max}+1$). However, averaging these functions for each value of t over a large number of avalanches, a typical shape emerges. These averages give rise to the functions $A(t)$ describing the area growth rate and $S(t)$, which is the size growth rate of the avalanche. According to the dynamic scaling assumption, each one of these functions can be written in the general scaling form,

$$X(t)=K_X\langle t\rangle_X^{-\alpha_X} f_X(\mu), \qquad (6)$$

where $\mu=t/\langle t\rangle_X$, $X\in\{S,A\}$, and

$$\langle t\rangle_X=\frac{\sum_t tX(t)}{\sum_t X(t)}. \qquad (7)$$

The scaling function $f_X(\mu)$ satisfies the sum rules

$$\int_0^\infty f_X(\mu)d\mu=\int_0^\infty \mu f_X(\mu)d\mu=1. \qquad (8)$$

The scaling functions $f_S(\mu)$ and $f_A(\mu)$ for the BTW and the Manna models were introduced in Ref. [26], based on simulation results for $L=128$, 256, and 512. It was shown that the scaling functions for these two models are different from each other, supporting the existence of two universality classes. Here we examine the universality of these functions within the classes of deterministic Abelian and stochastic models. The scaling functions obtained for models in these two classes are shown in Fig. 3. In the deterministic Abelian class we observe a good coincidence between the BTW and the Abelian Zhang models for both $f_S(\mu)$ and $f_A(\mu)$. Similarly, in the stochastic class we observe a good coincidence between the Manna and stochastic Zhang models. These results provide further evidence for universality within each of the two classes.

To characterize the avalanche structures we examine the function $f(\mathbf{i})$, that provides the number of toppling events at site $\mathbf{i}$ during the avalanche. The function $f(\mathbf{i})$ is shown for the BTW model [Fig. 4(a)], the abelian Zhang model [Fig. 4(b)], the Manna model [Fig. 4(c)], and the stochastic Zhang model [Fig. 4(d)]. For the deterministic models, we observe a shell structure in which all sites that relaxed at least $n+1$ times form a connected cluster with no holes, which is contained in the cluster of sites that relaxed at least n times [11]. The Stochastic models exhibit a random avalanche structure with many peaks and holes [25,26]. A related and useful concept in the analysis of the avalanche structure in deterministic Abelian models is the decomposition of an avalanche into *waves* of topplings [37–40]. The waves are obtained by a specific ordering of the toppling events (after energy δE was added at site $\mathbf{i}$). The first wave consists of the toppling of all the sites that became unstable due to the first toppling of $\mathbf{i}$, before $\mathbf{i}$ is toppled for the second time. Similarly, the nth wave is generated after the nth toppling of $\mathbf{i}$, until at the end of the last wave $\mathbf{i}$ remains stable. The wave

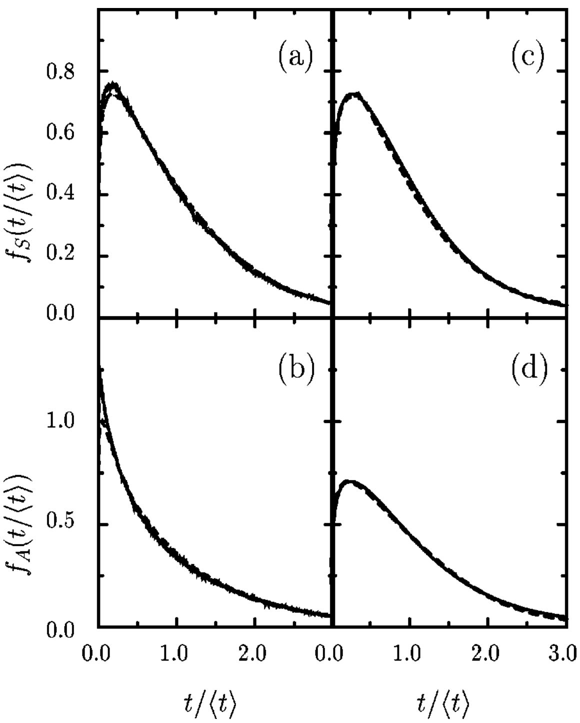

FIG. 3. The scaling functions for the Abelian and stochastic models. (a) $f_S(t/\langle t\rangle)$ for the stochastic models, Manna (—) and stochastic Zhang (- -); (b) $f_A(t/\langle t\rangle)$ for the same stochastic models; (c) $f_S(t/\langle t\rangle)$ for the Abelian models, BTW (—) and Abelian Zhang (- -); (d) $f_A(t/\langle t\rangle)$ for the same Abelian models. In (a) and (b) we observe a good coincidence between the scaling function for the Manna and the stochastic Zhang models, while in (c) and (d) we observe a good coincidence between the BTW and the Abelian Zhang models.

structure is simpler than the entire avalanche since, for deterministic Abelian models, each site can topple at most once during a wave, which enables complete analysis of their scaling properties [40]. The concept of waves does not seem to apply in the case of stochastic models, in which sites can topple multiple times between topplings of **i**.

As an additional test of universality we compare the distribution exponent τ_s between the two models in each class. In Fig. 5 we present on a log-log scale the distribution $P(s)$ vs s for the BTW (●) and the Abelian Zhang (□) models [Fig. 5(a)], and for the Manna (●) and the stochastic Zhang (□) models [Fig. 5(b)]. Good agreement is found in both cases. For the two deterministic models the slope of the straight line segment gives rise to $\tau_s = 2.10 \pm 0.02$. For the two stochastic models, $\tau_s = 2.25 \pm 0.02$ for the Manna model and $\tau_s = 2.23 \pm 0.02$ for the stochastic Zhang model. As noted in Ref. [25] the exponent τ_s is not the most reliable measure of universality due to its relatively strong size dependence.

Consider the class of deterministic models that are non-Abelian. It turns out that such models exhibit crossover behavior. For example, the exponent τ_s of the generalized Zhang model varies continuously as the parameter p is lowered from the Zhang value $\tau_s = 2.26 \pm 0.02$ at $p=1$ to the BTW values $\tau_s = 2.10 \pm 0.02$ as $p \to 0$ (Fig. 6). Also, for the Zhang model the conditional expectation value $E[s|a]$ vs a exhibits a bend typical of a crossover behavior [26]. The average size and area growth functions during the avalanche in the Zhang model do not collapse into scaling functions [26]. For the generalized Zhang model the avalanche structure is intermediate between those of the deterministic Abelian and stochastic models and depends on the parameter p. As $p \to 0$ the avalanche structure converges to the ordered shell structure of the BTW model [Fig. 4(b)].

We conclude that the universality class of deterministic sandpile models includes only the ones that are Abelian, such as the BTW and the Abelian Zhang models. The universality class of stochastic sandpiles includes both Abelian models (such as the Manna model) and non-Abelian models (such as the stochastic Zhang model).

IV. DISCUSSION

In recent years the universality in sandpile models was studied theoretically using the fixed scale transformation approach [19,20,41,42] as well as the dynamic renormalization group (RG) approach [21–23]. The approach introduced in Refs. [19,20] is a real space renormalization group scheme. It provides recursion equations for the dynamical variables as the system is coarse grained from smaller to larger cells, coupled to a stationary condition in order to maintain the critical state. The equations exhibit a single stable fixed point that is reached from either the BTW or the Manna parameters at the microscopic scale. The approach introduced in Refs. [23] is a dynamical RG scheme based on a nonlinear partial differential equation that is designed to describe the long wavelength behavior of the sandpile dynamics. This approach was applied to the BTW and the Zhang models concluding that they belong to the same universality class.

Numerical studies of sandpile models, based on an extended set of critical exponents, scaling functions, and other geometric features of the avalanches provided evidence that the BTW and Manna models belong to different universality classes [25,26]. Furthermore, it was recently found that the difference in scaling behavior of deterministic and stochastic models appears also in directed models [43].

To further examine the results of Refs. [25,26], the expectation values $E[s|a]$ for the BTW and Manna models were studied in Ref. [24] taking into account corrections to scaling. It was argued that since all the avalanches satisfy $s > a$, and the distribution $P(s|a)$ of sizes for avalanches of a given area is a broad asymmetric distribution, the scaling should be described according to

$$E[s|a] = a + Ca^{\gamma_{sa}}, \tag{9}$$

where C is a nonuniversal constant. Using this relation to fit the simulation results they obtain a linear fit with $\gamma_{sa} = 1.35 \pm 0.05$ for both models, suggesting that the two models belong to the same universality class. Note that in general the correction term could take the form a^{γ} where $\gamma < \gamma_{sa}$, so the choice of a^1 is somewhat arbitrary. The values of the

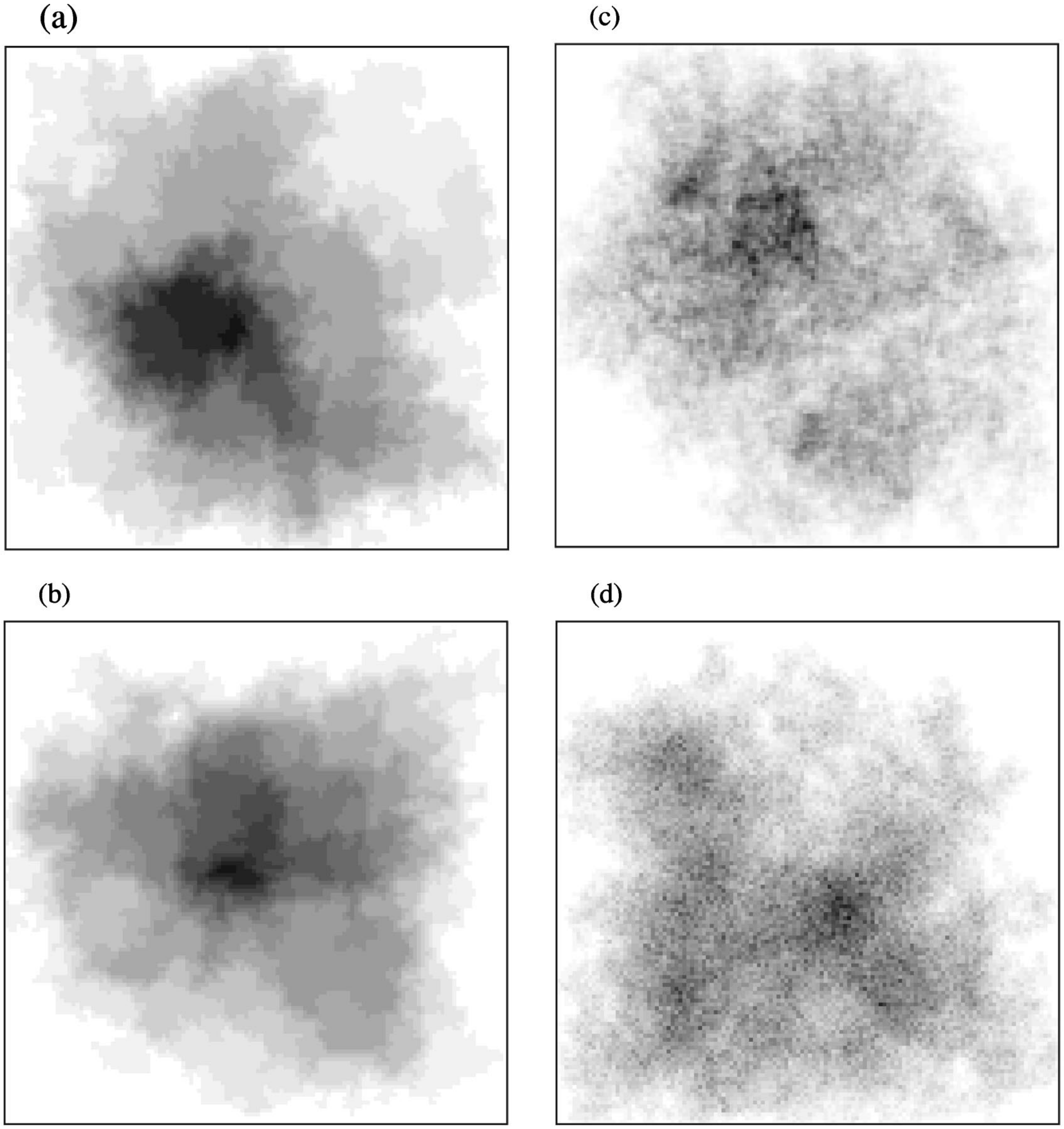

FIG. 4. Typical large avalanche structures for the BTW model (a), the Abelian Zhang model (b), the Manna two-state model (c), and the stochastic Zhang model (d). The gray scales indicate the number of toppling events that occurred at each site during the avalanche. White represents zero relaxations and black represents the most active domains with the maximal number of relaxations. The system size is $L = 128$. It is observed that both deterministic Abelian models exhibit a regular structure of domains of high activity inside domains of lower activity, while the stochastic models exhibit an irregular structure.

geometric exponents calculated previously, without the correction term, are $\gamma_{sa} = 1.06 \pm 0.01$, $\gamma_{at} = 1.53 \pm 0.01$, and $\gamma_{st} = 1.62 \pm 0.01$ for the BTW model and $\gamma_{sa} = 1.23 \pm 0.01$, $\gamma_{at} = 1.35 \pm 0.01$, and $\gamma_{st} = 1.70 \pm 0.01$ for the Manna model [25]. For both models, these values approximately satisfy the relation $\gamma_{sa}\gamma_{at} = \gamma_{st}$, while the value of γ_{sa} obtained in Ref. [24] does not satisfy this relation. This is due to the fact that for γ_{at} and γ_{st} one does not expect a correction term of this type. More recently, the conditional probability distribution $P(s|a)$ of the avalanche sizes for a given area was examined using moment analysis [27,28]. It was found to exhibit multifractal scaling indicating that the scaling behavior of this distribution cannot be fully characterized by a single exponent but requires a spectrum of critical exponents. The results of this more complete analysis confirm the conclusion of Refs. [25,26] that the BTW and Manna models belong to different universality classes.

Further evidence for the distinction between the BTW and the Manna models was obtained from the analysis of these models under closed (or periodic) boundary conditions [29–31]. In this case both the drive and the dissipation are removed and the total energy of the system is conserved. Under these conditions the models exhibit a second order dynamical phase transition between a static phase and a dynamical phase in which the avalanche is persistent. The critical exponents that characterize the phase transition were found to be different in the BTW and the Manna models, providing further evidence that they belong to different uni-

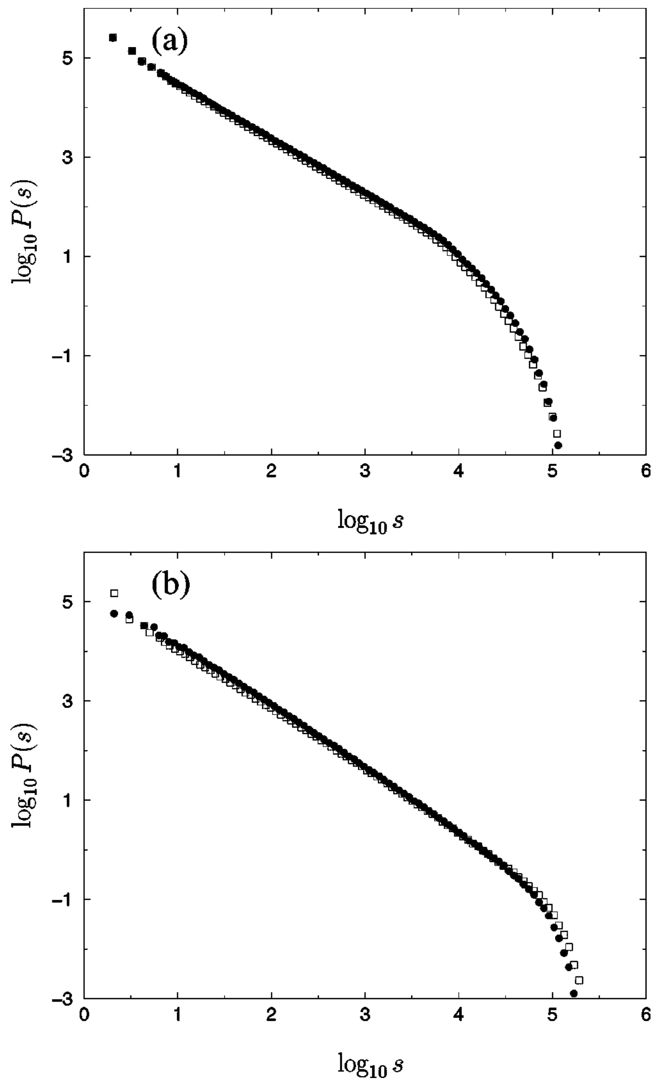

FIG. 5. The distribution of avalanche sizes $P(s)$ vs s: (a) for the BTW (●) and the Abelian Zhang (with $p=0.05$) models (□); (b) for the Manna (●) and the stochastic Zhang models (□) for $L=128$. Good agreement is found between the results for the two models in each class.

versality classes. Crossover phenomena between the two classes have also been studied [44].

In this paper we focused on conservative, undirected sandpile models with short range dynamic rules. There is evidence that all three properties are relevant and therefore our results apply only within this type of models. For nonconservative models it was found that the critical exponents are nonuniversal. They depend on a parameter that determines the rate at which energy is dissipated [45,46]. Directed sandpile models were found to exhibit a universality class of deterministic models [4] as well as a universality class of stochastic models [47,43,48], both exhibit different critical behavior than the corresponding classes of undirected models.

It was recently shown that models of Eulerian walkers exhibit SOC [49,50]. These are deterministic models that share the same operator algebra as the Abelian sandpile models, however the relaxation mechanism in these models is different. Their critical properties are found to be different than those of sandpile models.

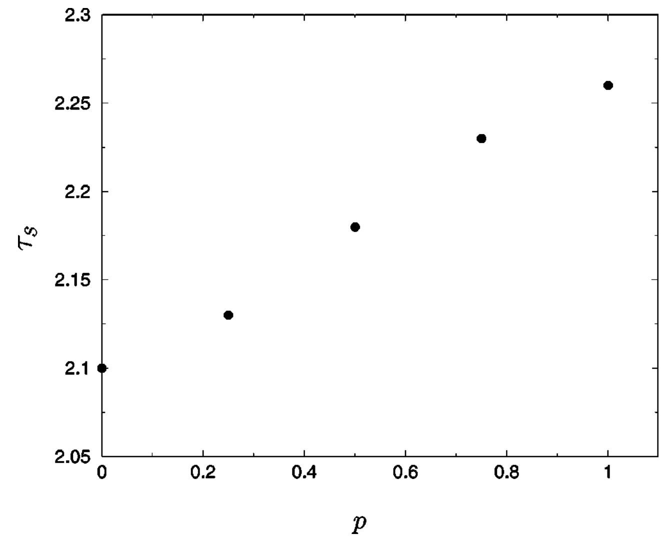

FIG. 6. The exponent τ_s of the avalanche size distribution for the generalized Zhang model as a function of the parameter $0<p\leqslant 1$. It is found that τ_s varies smoothly as a function of p. The model coincides with the Zhang model for $p=1$, where $\tau_s=2.26\pm 0.02$. As p is lowered, τ_s decreases towards the value for the deterministic Abelian models, namely, $\tau_s=2.10\pm 0.02$ in the limit $p\rightarrow 0$. The system size is $L=128$.

V. SUMMARY AND CONCLUSIONS

Recent numerical studies of conservative, undirected sandpile models with short range dynamic rules provided evidence that deterministic Abelian models such as the BTW model and stochastic models such as the Manna model exhibit different scaling behavior and thus do not belong to the same universality class. In this paper we examined the universality within each of these two classes of models, namely, the deterministic Abelian class and the stochastic class. To this end we considered additional models in the two classes. We then used an extended set of critical exponents, scaling functions, and geometric features to characterize the avalanche dynamics of the different models in two dimensions. Comparisons of the results show nearly identical scaling properties for the models within the deterministic Abelian class as well as within the stochastic class, providing evidence for universality within each of these two classes. Significant differences are observed between the two classes, in agreement with previous studies [25,26], indicating that these are two distinct universality classes. It was also observed that deterministic models that are non-Abelian exhibit critical exponents that depend on a parameter, namely, they are nonuniversal.

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